Exam Advanced Mechanics, Wednesday, April 12, 2017

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5 problems (total of 47 points).
The solution of every problem on a separate piece of paper with name and student number.
Some useful formulas are listed at the end.

Problem 1 (11 pnts in total)

Consider the eom of a damped oscillator which is driven by a time-dependent force, $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F(t)/m$.

2 pnts a. Give the complete expression for the Greens function $G(t,t')$ for the oscillator of this problem. Hint: look at the formulas at the end of this exam.

2 pnts b. Show, by direct substitution in the eom that the Greens function solves the eom for $F(t) = 0$ at all times except $t = t'$.

2 pnts c. Give the explicit expression for $x(t)$ in terms of a definite integral for the case that

$$F(t) = \begin{cases} 0 & t < -\tau/2 \\ (a/\tau)t & -\tau/2 < t < \tau/2 \\ 0 & t > \tau/2 \end{cases} .$$

Distinguish three cases: $t < -\tau/2$, $-\tau/2 < t < \tau/2$, and $t > \tau/2$.

3 pnts d. Solve for $x(t)$ for $t > \tau/2$ in the limit $\beta = 0$.

2 pnts e. What do you expect for $x(t)$ in the limit $\omega_0\tau \ll 1$ and give a short explanation.

Problem 2 (10 pnts in total)

A particle of mass m is moving in a potential given by

$$V(r) = \frac{g}{r^n} .$$

1 pnts a. Write Lagrangian for this problem.

2 pnts b. Give the constants of motion.

2 pnts c. Show that the Euler-Lagrange equation for the radial motion for this problem can be expressed as

$$m\ddot{r} - \frac{l^2}{mr^3} - \frac{ng}{r^{n+1}} = 0 .$$

2 pnts d. Assume that the particle is moving in a circular orbit. Give the expression for the radius of the circular orbit r_0.

1 pnts e. Calculate the orbiting frequency ω_0 for a particle in the circular orbit.

2 pnts f. The motion of the particle is almost a circular orbit. Calculate the frequency of small oscillations around the circular orbit.

Problem 3 (8 pnts in total)

A coin is thrown up in free space, rotating with an angular frequency $\vec{\omega}$ around an arbitrary axis. The momenta of inertia along its principal axes are $I_1 = I_2$ and I_3.

2 pnts a. Which one is larger, I_1 or I_3? Give a short motivation.

1 pnts b. Give the expression for the components of the angular momentum vector, $\vec{L}$, in the body-fixed frame.

2 pnts c. The equations of motion are given by $d\vec{L}/dt = \vec{N} = 0$. Express this in terms of the momenta of inertia and the components of $\vec{\omega}$ along the principal axes and their time derivatives.

2 pnts d. Show that $\omega_1(t) = A\cos\beta t$, $\omega_2(t) = A\sin\beta t$, and $\omega_3(t) = B$, is a solution of the equations of motion and give the expression for β.

1 pnts e. Which vector remains fixed in the inertial system.

Problem 4 new (9 pnts in total)

The vector potential of a plane-wave photon field is given by $A^\mu(x) = C^\mu e^{ikx}$ where $kx = k^\nu x_\nu$. For this particular problem we use $C^\mu = (0,1,0,0)$ and $k^\mu = \omega(1,0,0,1)$.

2 pnts a. Evaluate $\partial_\mu kx$.

1 pnts b. Evaluate $\partial^\mu kx$.

1 pnts c. Evaluate $\partial_\mu e^{ikx}$.

3 pnts d. Show that $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ can be expressed as $F^{\mu\nu} = ik^\mu A^\nu - ik^\nu A^\mu$.

2 pnts e. Show that $\partial_\mu F^{\mu\nu} = 0$.

Problem 5 new (9 pnts in total)

A particle with charge e and rest-mass m is moving in a uniform electric field in the z-direction, $\vec{E} = E_0\hat{z}$. There is no gravitational interaction to consider. At time $t = 0$ the particle has a momentum $\vec{p}(t) = p_0\hat{x}$.

2 pnts a. Construct a four-vector potential for this problem for which $\alpha = \partial^\mu A_\mu = 0$.

2 pnts b. Construct a four-vector potential for this problem for which $\alpha = \partial^\mu A_\mu \neq 0$.

2 pnts c. Use the equations of motion for this problem, $\frac{d}{dt}\vec{p} = e\vec{E}$, to obtain an expression for the time-derivatives of the components of the momenta of the particle.

2 pnts d. Derive an equation for the time-dependence of total energy $\mathcal{E}$.

1 pnts e. Use the previous result to show that the velocity in the $\hat{x}$-direction decreases in time.

Possibly useful formulas:

$\vec{F}_B = \vec{F}_{\text{inert}} - 2m\vec{\omega} \times \vec{v}_B - m\dot{\vec{\omega}} \times \vec{r}_B - m\vec{\omega} \times (\vec{\omega} \times \vec{r}_B)$, and $\vec{v}_I = \vec{v}_B + \vec{\omega} \times \vec{r}_B$

The response of a damped oscillator $\ddot{x} + 2\beta\dot{x} + \omega_r^2 x = F(t)/m$ to a delta force at $t = 0$ is $\frac{1}{\omega_1 m} e^{-\beta t} \sin \omega_1 t$ for $t > 0$, where $\omega_1 = \sqrt{\omega_r^2 - \beta^2}$.

The 'alternative' form of the Euler equation for $f(y, y'; x)$ is

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right)$$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$; $\cos(\alpha - \beta) = \sin\alpha\sin\beta + \cos\alpha\cos\beta$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}$

$\vec{B} = \vec{\nabla} \times \vec{A}$; $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial ct}$

Integrals

For $c > 0$ we have:

$$\int e^{cx} dx = \frac{1}{c} e^{cx} \; ; \quad \int x\, e^{cx} dx = \frac{cx - 1}{c^2} e^{cx} \; ; \quad \int x^2 e^{cx} dx = \frac{c^2x^2 - 2cx + 2}{c^3} e^{cx}$$